REFINEMENTS OF SOME REVERSES OF SCHWARZ'S INEQUALITY IN 2−INNER PRODUCT SPACES AND APPLICATIONS FOR INTEGRALS

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Abstract. Refinements of some recent reverse inequalities for the celebrated Cauchy-Bunyakovsky-Schwarz inequality in 2−inner product spaces are given. Using this framework, applications for determinantal integral inequalities are also provided.

1. Introduction

The concepts of 2−inner products and 2−inner product spaces have been intensively studied by many authors in the last three decades.

A systematic presentation of the recent results related to the theory of 2−inner product spaces as well as an extensive list of the related references can be found in the book [5]. We recall here the basic definitions and the elementary properties of 2−inner product spaces that will be used in the sequel (see also [3]).

Let X be a linear space of dimension greater than 1 over the number field $\mathbb{K}$, when $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2I_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x, x|z) = (z, z|x)$,
($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,
($2I_4$) $(\alpha x, y|z) = \alpha (x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,
($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$,

where $x, x', y, z \in X$. The functional $(\cdot, \cdot|\cdot)$ is called a 2−*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2−*inner product space* (or *2-pre-Hilbert space*) [5].

Some basic properties of the 2−inner product spaces can be immediately obtained as follows:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2I_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2I_3$) and ($2I_4$), we have

$$(0, y|z) = (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha} (x, y|z). \tag{1.1}$$

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(3) Using $(2I_3)-(2I_5)$, we have

$$\begin{aligned}(z,z|x\pm y) &= (x\pm y,x\pm y|z)\\ &= (x,x|z)+(y,y|z)\pm 2\operatorname{Re}(x,y|z)\end{aligned}$$

and

$$\operatorname{Re}(x,y|z)=\frac{1}{4}\left[(z,z|x+y)-(z,z|x-y)\right]. \tag{1.2}$$

In the real case $\mathbb{K}=\mathbb{R}$, (1.2) reduces to

$$(x,y|z)=\frac{1}{4}\left[(z,z|x+y)-(z,z|x-y)\right], \tag{1.3}$$

and using this formula, it is easy to see, for any $\alpha\in\mathbb{R}$, that

$$(x,y|\alpha z)=\alpha^2(x,y|z). \tag{1.4}$$

In the complex case, $\mathbb{K}=\mathbb{C}$, using (1.1) and (1.2), we have

$$\operatorname{Im}(x,y|z)=\operatorname{Re}\left[-i(x,y|z)\right]=\frac{1}{4}\left[(z,z|x+iy)-(z,z|x-iy)\right],$$

which, in combination with (1.2), yields

$$(x,y|z)=\frac{1}{4}\left[(z,z|x+y)-(z,z|x-y)\right]+\frac{i}{4}\left[(z,z|x+iy)-(z,z|x-iy)\right]. \tag{1.5}$$

Using (1.5) and (1.1), we have, for any $\alpha\in\mathbb{C}$, that

$$(x,y|\alpha z)=|\alpha|^2(x,y|z). \tag{1.6}$$

However, for $\alpha\in\mathbb{R}$, (1.6) reduces to (1.4). Also, from (1.6) it follows that

$$(x,y|0)=0.$$

(4) For any three given vectors $x,y,z\in X$, consider the vector $u=(y,y|z)x-(x,y|z)y$. By $(2I_1)$, we know that $(u,u|z)\geq 0$ with the equality if and only if u and z are linearly dependent. It is obvious that the inequality $(u,u|z)\geq 0$ can be rewritten as

$$(y,y|z)\left[(x,x|z)(y,y|z)-|(x,y|z)|^2\right]\geq 0. \tag{1.7}$$

For $x=z$, (1.7) becomes

$$-(y,y|z)|(z,y|z)|^2\geq 0$$

which implies that

$$(z,y|z)=(y,z|z)=0, \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) also holds.

Now, if y and z are linearly independent, then $(y,y|z)>0$, and from (1.7), it follows the Cauchy-Bunyakovsky-Schwarz inequality ($CBS-$inequality for short) for $2-$inner products:

$$|(x,y|z)|^2\leq (x,x|z)(y,y|z). \tag{1.9}$$

Utilizing (1.8), it is easy to see that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x,y,z\in X$ and is strict unless the vectors

$$u=(y,y|z)x-(x,y|z)y \text{ and } z$$

are linearly dependent. In fact, *we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent* [3].

In any given 2−inner product space $(X, (\cdot,\cdot|\cdot))$, we can define a function $\|\cdot|\cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x,x|z)} \tag{1.10}$$

for all $x, z \in X$. It is easy to see that, this function satisfies the following conditions

$(2N_1)$ $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,
$(2N_2)$ $\|z|x\| = \|x|z\|$,
$(2N_3)$ $\|\alpha x|z\| = |\alpha| \, \|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,
$(2N_4)$ $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the conditions $(2N_1) - (2N_4)$ is called a 2-*norm* on X and $(X, \|\cdot|\cdot\|)$ is called a *linear 2-normed space* [9].

In terms of 2-norms, the (CBS) −inequality (1.9) can be written as

$$|(x,y|z)|^2 \leq \|x|z\|^2 \|y|z\|^2. \tag{1.11}$$

The equality in (1.11) holds if and only if x, y and z are linearly dependent.

For recent inequalities in 2-inner products, see the recent works [1] - [13] and the references therein.

In [7], the authors pointed out the following reverses of the (CBS) −inequality in 2-inner product spaces.

Assume that $x, y, z \in X$ and $a, A \in \mathbb{K}$ are such that either

$$\operatorname{Re}(Ay - x, x - ay|z) \geq 0 \tag{1.12}$$

or, equivalently

$$\left\| x - \frac{a+A}{2}, y \Big| z \right\| \leq \frac{1}{2} |A - a| \, \|y|z\| \tag{1.13}$$

hold. Then one has the inequality [7]

$$0 \leq \|x|z\|^2 \|y|z\|^2 - |(x,y|z)|^2 \leq \frac{1}{4} |A - a|^2 \|y|z\|^4. \tag{1.14}$$

The constant $\frac{1}{4}$ is sharp in (1.14) in the sense that it cannot be replaced by a smaller constant.

With the same assumptions for x, y, z, a and A and, if moreover $\operatorname{Re}(\bar{a}A) > 0$, then [7]

$$\begin{aligned} \|x|z\| \, \|y|z\| &\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a}\right)(x,y|z)\right]}{\operatorname{Re}\left[(\bar{a}A)\right]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{|A + a|}{\operatorname{Re}\left[(\bar{a}A)\right]^{\frac{1}{2}}} |(x,y|z)|. \end{aligned} \tag{1.15}$$

Here the constant $\frac{1}{2}$ is best possible in both inequalities.

As a consequence of (1.15) we may get the following additive reverse of the (CBS) −inequality as well [7]

$$0 \leq \|x|z\|^2 \|y|z\|^2 - |(x,y|z)|^2 \leq \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(\bar{a}A)} |(x,y|z)|^2. \tag{1.16}$$

The constant $\frac{1}{4}$ in (1.16) is best possible in the above sense.

2. Refinements of a Reverse (CBS) −Inequality

The following reverse of the (CBS) −inequality holds.

Theorem 1. *Let $(X,(\cdot,\cdot|\cdot))$ be a 2-inner product space on $\mathbb{K}$, $x,y,z\in X$ and $a,A\in\mathbb{K}$. If*

$$\operatorname{Re}(Ay-x,x-ay|z)\geq 0, \tag{2.1}$$

or, equivalently,

$$\left\|x-\frac{a+A}{2}y|z\right\|\leq\frac{1}{2}\left|A-a\right|\left\|y|z\right\|, \tag{2.2}$$

holds, then one has the inequality

$$\begin{aligned} 0 &\leq \left\|x|z\right\|^2\left\|y|z\right\|^2-\left|(x,y|z)\right|^2 \\ &\leq \frac{1}{4}\left|A-a\right|^2\left\|y|z\right\|^4-\left|\frac{a+A}{2}\left\|y|z\right\|^2-(x,y|z)\right|^2 \\ &\quad\left(\leq\frac{1}{4}\left|A-a\right|^2\left\|y|z\right\|^4\right). \end{aligned} \tag{2.3}$$

The constant $\frac{1}{4}$ is sharp in (2.3) in the sense that it cannot be replaced by a smaller constant.

Proof. Observe , for $x,u,U\in X$, that we have

$$\begin{aligned}\frac{1}{4}\left\|U-u|z\right\|^2-\left\|x-\frac{u+U}{2}\right|z\right\|^2 &= \operatorname{Re}(U-u,x-u|z) \\ &= \operatorname{Re}\left[(x,u|z)+(U,x|z)\right]-\operatorname{Re}(U,u|z)-\left\|x,z\right\|^2.\end{aligned}$$

Therefore

$$\operatorname{Re}(U-u,x-u|z)\geq 0,$$

if and only if

$$\left\|x-\frac{u+U}{2}\right|z\right\|\leq\frac{1}{2}\left\|U-u|z\right\|.$$

If we choose above $U=Ay$ and $u=ay$, we deduce that the conditions (2.1) and (2.3) are equivalent.

Now, if we consider $x,y,z\in X$ and $\lambda\in\mathbb{K}$, then we may state that

$$\left\|x-\lambda y|z\right\|^2=\left\|x|z\right\|^2-2\operatorname{Re}\left[\lambda(x,y|z)\right]+\left|\lambda\right|^2\left\|y|z\right\|^2 \tag{2.4}$$

and

$$\left|\lambda\left\|y|z\right\|^2-(x,y|z)\right|^2=\left|\lambda\right|^2\left\|y|z\right\|^2-2\left\|y|z\right\|^2\operatorname{Re}\left[\lambda(x,y|z)\right]+\left|(x,y|z)\right|. \tag{2.5}$$

If we multiply (2.4) by $\left\|x|z\right\|^2\geq 0$ and then subtract equation (2.5), we deduce the following equality, that is of interest in itself,

$$\left\|x|z\right\|^2\left\|y|z\right\|^2-\left|(x,y|z)\right|^2=\left\|x-\lambda y|z\right\|^2\left\|y|z\right\|^2-\left|\lambda\left\|y|z\right\|^2-(x,y|z)\right|^2. \tag{2.6}$$

If we now use (2.6) for $\lambda=\frac{a+A}{2}$ and take into account (2.2), then we deduce the desired inequality (2.3).

To prove the sharpness of the constant $\frac{1}{4}$ in the second inequality in (2.3), assume that, this inequality holds with a constant $C > 0$. That is,

$$\|x|z\|^2 \|y|z\|^2 - |(x,y|z)|^2 \le C|A-a|^2 \|y|z\|^4 - \left| \frac{a+A}{2} \|y|z\|^2 - (x,y|z) \right|^2, \tag{2.7}$$

where x, y, z, a and A satisfy the hypothesis of the theorem.

Consider $y, z \in X$ with $\|y|z\| = 1$, $a \neq A$, $a, A \in \mathbb{K}$ and $m \in X$ with $\|m|z\| = 1$ and $(y, m|z) = 0$. Define the vector

$$x := \frac{a+A}{2} y + \frac{A-a}{2} m.$$

Then a simple calculation shows that

$$(Ay - x, x - ay|z) = \frac{|A-a|^2}{4} (y - m, y + m|z) = 0,$$

and thus the condition (2.1) is fulfilled.

Observe also that

$$\|x|z\|^2 = \left\| \frac{a+A}{2} y + \frac{A-a}{2} m \middle| z \right\| = \left| \frac{A+a}{2} \right|^2 + \left| \frac{A-a}{2} \right|^2,$$

and

$$(x, y|z) = \left(\frac{a+A}{2} y + \frac{A-a}{2} m, y|z \right) = \frac{a+A}{2}.$$

Consequently, by (2.7), we deduce

$$\frac{(A-a)^2}{4} \le C|A-a|^2,$$

giving $C \ge \frac{1}{4}$, and the theorem is proved. ∎

Another reverse for the (CBS)-inequality is incorporated in the following theorem.

Theorem 2. *With the assumptions of Theorem 1, one has the inequality*

$$\begin{aligned} 0 &\le \|x|z\|^2 \|y|z\|^2 - |(x,y|z)|^2 \\ &\le \frac{1}{4} |A-a|^2 \|y|z\|^4 - \mathrm{Re}\,(Ay - x, x - ay|z) \|y|z\|^2 \\ &\quad \left(\le \frac{1}{4} |A-a|^2 \|y|z\|^4 \right). \end{aligned} \tag{2.8}$$

The constant $\frac{1}{4}$ is sharp in (2.8).

Proof. We use the following identity that has been obtained in [7] and can be proved by direct computation

$$\begin{aligned} &\|x|z\|^2 \|y|z\|^2 - |(x,y|z)|^2 \\ &\quad = \mathrm{Re}\left[\left(A\|y|z\|^2 - (x,y|z) \right) \left(\overline{(x,y|z)} - \bar{a}\|y|z\|^2 \right) \right] \\ &\qquad - \|y|z\|^2 \,\mathrm{Re}\,(Ay - x, x - ay|z). \end{aligned} \tag{2.9}$$

By the elementary inequality

$$\mathrm{Re}\left(\alpha\bar{\beta}\right) \le \frac{1}{4} |\alpha + \beta|^2, \qquad \alpha, \beta \in \mathbb{K}$$

applied for

$$\alpha := A\|y|z\|^2 - (x,y|z) \quad \text{and} \quad \beta = (x,y|z) - a\|y|z\|^2,$$

we deduce the required inequality (2.8).

The sharpness of the constant may be proved as above in Theorem 1 and we omit the details. ∎

3. Another Reverse for the (CBS) –Inequality

The following result also holds.

Theorem 3. *Let $(X;(\cdot,\cdot|\cdot))$ be a 2-inner product space over $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $x,y,z\in X$, $a,A\in\mathbb{K}$. If $A\neq -a$ and either*

$$\operatorname{Re}(Ay - x, x - ay|z) \geq 0 \tag{3.1}$$

or, equivalently,

$$\left\| x - \frac{a+A}{2} y \right| z \Big\| \leq \frac{1}{2}|A-a|\,\|y|z\|, \tag{3.2}$$

holds, then we have the inequality

$$\begin{aligned} 0 &\leq \|x|z\|\,\|y|z\| - \operatorname{Re}\left[\operatorname{sgn}\left(\frac{a+A}{2}\right)(x,y|z)\right] \\ &\leq \|x|z\|\,\|y|z\| - |(x,y|z)| \\ &\leq \frac{1}{4}\frac{|A-a|^2}{|A+a|}\|y|z\|^2, \end{aligned} \tag{3.3}$$

where $\operatorname{sgn}(\alpha) := \frac{\alpha}{|\alpha|}$, $\alpha\in\mathbb{C}\setminus\{0\}$.

The $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. We observe that the condition (3.2) is equivalent with

$$\|x|z\|^2 - 2\operatorname{Re}\left[\left(\frac{a+A}{2}\right)(x,y|z)\right] + \left|\frac{a+A}{2}\right|^2\|y|z\|^2 \leq \frac{1}{4}|A-a|^2\|y|z\|^2$$

giving

$$\begin{aligned} \|x|z\|^2 + \left|\frac{a+A}{2}\right|^2\|y|z\|^2 &\leq \frac{1}{4}|A-a|^2\|y|z\|^2 + 2\operatorname{Re}\left[\left(\frac{a+A}{2}\right)(x,y|z)\right] \\ &\leq \frac{1}{4}|A-a|^2\|y|z\|^2 + 2\left|\frac{a+A}{2}\right||(x,y|z)|. \end{aligned} \tag{3.4}$$

By the elementary inequality

$$\alpha^2+\beta^2 \geq 2\alpha\beta, \quad \alpha,\beta\geq 0,$$

we have

$$2\left|\frac{a+A}{2}\right|\|x|z\|\,\|y|z\| \leq \|x|z\|^2 + \left|\frac{a+A}{2}\right|^2\|y|z\|^2. \tag{3.5}$$

By making use of (3.4) and (3.5), we deduce

$$\begin{aligned} 0 &\leq \left|\frac{a+A}{2}\right| \|x|z\| \, \|y|z\| - \operatorname{Re}\left[\left(\frac{a+A}{2}\right)(x,y|z)\right] \\ &\leq \left|\frac{a+A}{2}\right| \left[\|x|z\| \, \|y|z\| - |(x,y|z)|\right] \\ &\leq \frac{1}{8}\left|A-a\right|^2 \|y|z\|^2 , \end{aligned}$$

which is clearly equivalent to the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{4}$ in (3.3), let us assume that there is a constant $D > 0$ such that

$$\|x|z\| \, \|y|z\| - |(x,y|z)| \leq D \cdot \frac{|A-a|^2}{|A+a|} \|y|z\|^2 , \tag{3.6}$$

provided x, y, z and a, A satisfy the hypotheses of the theorem.

Assume now, $x, y, z, e \in X$ are such that $\|y, z\| = 1$, $\|e, z\| = 1$ and $(e, y|z) = 0$. For $a, A \in \mathbb{K}$ with $a \neq -A$, define

$$x = \frac{a+A}{2} y + \frac{A-a}{2} e.$$

Then

$$\left\| x - \frac{a+A}{2}, y|z \right\| = \frac{1}{2}\left|A-a\right| ,$$

and thus the condition (3.2) is satisfied with equality.

Observe that, with the above choices for x, y, z and e we have

$$\begin{aligned} \|x|z\| &= \sqrt{\frac{|A+a|^2}{4} + \frac{|A-a|^2}{4}} = \sqrt{\frac{|A|^2 + |a|^2}{2}}, \\ |(x,y|z)| &= \left|\frac{a+A}{2}\right|, \end{aligned}$$

and thus, from (3.6), we deduce the inequality

$$\sqrt{\frac{|A|^2 + |a|^2}{2}} - \left|\frac{a+A}{2}\right| \leq D \cdot \frac{|A-a|^2}{|A+a|} \tag{3.7}$$

for $a, A \in \mathbb{C}$, $a \neq -A$.

For $\varepsilon \in (0,1)$, consider $A = 1 + \sqrt{\varepsilon}$, $a = 1 - \sqrt{\varepsilon}$. Then $a \neq -A$ and by (3.9) we deduce

$$\sqrt{1+\varepsilon} - 1 \leq 2D\varepsilon,$$

giving by multiplication by $\sqrt{1+\varepsilon} + 1 > 0$ that

$$\varepsilon \leq 2\varepsilon\left(\sqrt{1+\varepsilon} + 1\right) D.$$

Since $\varepsilon \in (0,1)$, we may divide by ε and thus we get

$$D \geq \frac{1}{2\left(\sqrt{1+\varepsilon} + 1\right)}, \quad \varepsilon \in (0,1). \tag{3.8}$$

Letting $\varepsilon \to 0+$ in (3.8), we obtain $D \geq \frac{1}{4}$, and the sharpness of the constant is proved. ∎

When the constants A, a are real, we can point out the following reverse of the triangle inequality.

Corollary 1. *Let $(X; (\cdot, \cdot|\cdot))$ be a 2-inner product space over $\mathbb{K}$, $x, y, z \in X$, and $m, M \in (0, \infty)$ with $M > m$. If either*

$$\operatorname{Re}(My - x, x - my|z) \geq 0 \tag{3.9}$$

or, equivalently,

$$\left\| x - \frac{m+M}{2} y \right| z \right\| \leq \frac{1}{2}(M - m)\|y|z\| \tag{3.10}$$

holds, then we have the inequality

$$0 \leq \|x|z\| + \|y|z\| - \|x + y|z\| \leq \frac{1}{2} \cdot \frac{(M-m)}{\sqrt{M+m}} \|y|z\|. \tag{3.11}$$

Proof. A simple computation shows that

$$(\|x|z\| + \|y|z\|)^2 - \|x + y|z\|^2 = 2(\|x|z\| \|y|z\| - \operatorname{Re}(x, y|z)).$$

Using the inequality (3.3), we may state that

$$(\|x|z\| + \|y|z\|)^2 \leq \|x + y|z\|^2 + \frac{1}{4}\frac{(M-m)^2}{(M+m)}\|y|z\|^2. \tag{3.12}$$

Taking the square root of (3.12), we get

$$\begin{aligned} \|x|z\| + \|y|z\| &\leq \sqrt{\|x + y|z\|^2 + \frac{1}{4}\frac{(M-m)^2}{(M+m)}\|y|z\|^2} \\ &\leq \|x + y|z\| + \frac{1}{2} \cdot \frac{(M-m)}{\sqrt{M+m}}\|y|z\| \end{aligned}$$

and the inequality (3.11) is proved. ∎

Remark 1. *Firstly, let us observe that from the inequality (1.15) in the Introduction, we may state the following additive reverse of the $(CBS)-$inequality*

$$0 \leq \|x|z\| \|y|z\| - |(x, y|z)| \leq \frac{1}{2} \cdot \frac{|A + a| - 2[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}}{[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}} |(x, y|z)|, \tag{3.13}$$

provided $x, y, z \in X$, $a, A \in \mathbb{K}$ with $\operatorname{Re}(A\bar{a}) > 0$ and either the condition (2.1) or, equivalently (2.2), is valid.

If $M > m > 0$ and either (3.9) or, equivalently, (3.10) holds, then from (3.13) we may state the following simpler form

$$0 \leq \|x|z\| \|y|z\| - |(x, y|z)| \leq \frac{1}{2} \cdot \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{Mm}} |(x, y|z)|. \tag{3.14}$$

If, for the same M, m we write the inequality (3.3), then we have another bound, namely:

$$0 \leq \|x|z\| \|y|z\| - |(x, y|z)| \leq \frac{1}{4} \cdot \frac{(M-m)^2}{(M+m)} \|y|z\|^2, \tag{3.15}$$

provided (3.9), or equivalently, (3.10) holds.

4. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of parts of Ω and a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$, the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω. That is,

$$\int_\Omega \rho(t) |f(s)|^2 d\mu(s) < \infty,$$

where $\rho : \Omega \to (0, \infty)$ is a measurable function on Ω.

If we denote by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \qquad a, b, c, d \in \mathbb{R}$$

the determinant associated with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R};$$

then we can introduce on $L^2_\rho(\Omega)$ the following 2-inner product

$$(f, g|h)_\rho := \frac{1}{2}\int_\Omega \int_\Omega \rho(x)\rho(y) \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix} \times \begin{vmatrix} g(x) & g(y) \\ h(x) & h(y) \end{vmatrix} d\mu(x) d\mu(y), \tag{4.1}$$

generating the 2-norm

$$\|f|h\|_\rho = \left(\frac{1}{2}\int_\Omega \int_\Omega \rho(x)\rho(y) \begin{vmatrix} f(x) & f(y) \\ h(x) & h(y) \end{vmatrix}^2 d\mu(x) d\mu(y)\right)^{\frac{1}{2}}. \tag{4.2}$$

A simple computation with integrals shows that

$$(f, g|h)_\rho = \begin{vmatrix} \int_\Omega \rho(x) f(x) g(x) d\mu(x) & \int_\Omega \rho(x) f(x) h(x) d\mu(x) \\ \int_\Omega \rho(x) g(x) h(x) d\mu(x) & \int_\Omega \rho(x) h^2(x) d\mu(x) \end{vmatrix}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho(x) f^2(x) d\mu(x) & \int_\Omega \rho(x) f(x) h(x) d\mu(x) \\ \int_\Omega \rho(x) f(x) h(x) d\mu(x) & \int_\Omega \rho(x) h^2(x) d\mu(x) \end{vmatrix}^{\frac{1}{2}}.$$

We recall that the pair of functions $(q, p) \in L^2_\rho(\Omega) \times L^2_\rho(\Omega)$ is said to be *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for a.e. $x, y \in \Omega$.

Now, suppose that $h \in L_{\rho}^{2}(\Omega)$ is such that $h(x) \neq 0$ for a.e. $x \in \Omega$. Then by (4.1) we have the obvious identit,

$$(f,g|h)_{\rho} = \frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(x)\rho(y)h^{2}(x)h^{2}(y) \times\left(\frac{f(x)}{h(x)}-\frac{f(y)}{h(y)}\right)\left(\frac{g(x)}{h(x)}-\frac{g(y)}{h(y)}\right)d\mu(x)\,d\mu(y) \tag{4.3}$$

and thus, a *sufficient condition* for the inequality

$$(f,g|h)_{\rho} \geq 0 \tag{4.4}$$

to hold, is that the pair of functions $\left(\frac{f}{h},\frac{g}{h}\right)$ be synchronous. This condition is not necessary.

If $\Omega = [a,b] \subset \mathbb{R}$ $(a<b)$ and μ is the Lebesgue measure, then a sufficient condition for the functions $\left(\frac{f(x)}{h(x)},\frac{g(x)}{h(x)}\right)$, $x \in [a,b]$ to be synchronous is that they are monotonic in the same sense, i.e. $\frac{f}{h}$ and $\frac{f}{g}$ are both increasing or decreasing on $[a,b]$. Obviously, this condition is not necessary.

We are able now to state some integral inequalities that can be derived using the general framework presented above.

Proposition 1. *Let* $M > m > 0$ *and* $f,g,h \in L_{\rho}^{2}(\Omega)$, $h \neq 0$, *such that the functions*

$$M\cdot\frac{g}{h}-\frac{f}{h},\qquad \frac{f}{h}-m\cdot\frac{g}{h} \tag{4.5}$$

are synchronous on Ω*. Then we have the inequalities*

$$
\begin{aligned}
(4.6)\quad 0 &\leq \begin{vmatrix} \int_\Omega \rho f^2 & \int_\Omega \rho fh \\ \int_\Omega \rho fh & \int_\Omega \rho h^2 \end{vmatrix} \cdot \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho gh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix} \\
&\quad - \begin{vmatrix} \int_\Omega \rho fg & \int_\Omega \rho fh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix}^2 \\
&\leq \frac{1}{4}(M-m)^2 \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho gh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix}^2 \\
&\quad - \left| \frac{m+M}{2} \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho gh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix} - \begin{vmatrix} \int_\Omega \rho fg & \int_\Omega \rho fh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix} \right| \\
&\quad \left(\leq \frac{1}{4}(M-m)^2 \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho gh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix}^2 \right).
\end{aligned}
$$

The proof is obvious by Theorem 1 and we omit the details.

The following counterpart of the (CBS) $-$inequality for determinants also holds.

Proposition 2. *With the assumptions of Proposition 1, we have the inequality*

$$
\begin{aligned}
(4.7)\qquad 0 &\leq \begin{vmatrix} \int_\Omega \rho f^2 & \int_\Omega \rho fh \\ \int_\Omega \rho fh & \int_\Omega \rho h^2 \end{vmatrix} \cdot \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho gh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix} \\
&\quad - \begin{vmatrix} \int_\Omega \rho fg & \int_\Omega \rho fh \\ \int_\Omega \rho gh & \int_\Omega \rho h^2 \end{vmatrix}^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\frac{1}{4}(M-m)^2 \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix}\right. \\
&\qquad \left. - \begin{vmatrix} \int_\Omega (Mg-f)(f-mg) & \int_\Omega \rho (Mg-f) h \\ \int_\Omega \rho (f-mg) h & \int_\Omega \rho h^2 \end{vmatrix}\right) \\
&\qquad \times \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix} \\
&\qquad \left(\leq \frac{1}{4}(M-m)^2 \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix}^2\right).
\end{aligned}
$$

The proof follows by Theorem 2 applied for the 2-inner product defined in (4.3).

A different reverse of the (CBS) −inequality for determinants is incorporated in the following proposition.

Proposition 3. *With the assumptions of Proposition 1, we have the inequality*

$$
\begin{aligned}
0 &\leq \begin{vmatrix} \int_\Omega \rho f^2 & \int_\Omega \rho f h \\ \int_\Omega \rho f h & \int_\Omega \rho h^2 \end{vmatrix}^{\frac{1}{2}} \cdot \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix}^{\frac{1}{2}} \\
&\qquad - \left|\det \begin{pmatrix} \int_\Omega \rho f g & \int_\Omega \rho f h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{pmatrix}\right| \\
&\leq \frac{1}{4}\frac{(M-m)^2}{M+m} \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix}^2 .
\end{aligned} \tag{4.8}
$$

The constant $\frac{1}{4}$ is best possible in (4.8).

The proof follows from Theorem 3 applied for the 2-inner product defined in (4.3).

Finally, by the use of Corollary 1, we may state the following reverse of the triangle inequality for determinants.

Proposition 4. *With the assumptions of Proposition 1, we have the inequality:*

$$
\begin{aligned}
(4.9)\qquad 0 &\leq \begin{vmatrix} \int_\Omega \rho f^2 & \int_\Omega \rho f h \\ \int_\Omega \rho f h & \int_\Omega \rho h^2 \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} \int_\Omega \rho g h & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix}^{\frac{1}{2}} \\
&\quad - \begin{vmatrix} \int_\Omega \rho \left(f+g\right)^2 & \int_\Omega \rho \left(f+g\right) h \\ \int_\Omega \rho \left(f+g\right) h & \int_\Omega \rho h^2 \end{vmatrix}^{\frac{1}{2}} \\
&\leq \frac{1}{2} \cdot \frac{M-m}{\sqrt{M+m}} \cdot \begin{vmatrix} \int_\Omega \rho g^2 & \int_\Omega \rho g h \\ \int_\Omega \rho g h & \int_\Omega \rho h^2 \end{vmatrix}^{\frac{1}{2}}.
\end{aligned}
$$

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